

A neutral fermion model of 3 + 1 dimensional topological insulators

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We proposed an action of neutral fermions interacting with external electromagnetic fields, to construct the effective topological field theory of 3+1 dimensional topological insulators by integrating out the fermionic fields in the related path integral. These neutral quasiparticles are assumed to emerge from the collective behavior of the original physical particles and holes which establish the features of the space-time on which the theory is defined.

1 Introduction

Topological field theories by definition do not depend on the local features of the space-time in which they are defined. They appear in condensed matter physics as effective theories of quantum matter like topological insulators which are usually defined to be ordinary insulators in the bulk but conducting at the surface of the material (for a review see [1]). The material is under the influence of external electromagnetic fields. In the low energy limit, 3+1 dimensional topological insulators subject to electromagnetic fields given by the gauge fields A_μ ; $\mu = 0, 1, 2, 3$, are effectively described by the action ($\hbar = c = 1$),

$$S_{3D} = \frac{\theta e^2}{8\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma. \quad (1)$$

It is topological in the sense that the metric tensor of the related space-time manifold does not appear in the action and there is no local excitations. Properties of the underlying space-time should be fixed by additional information about the adopted microscopic model of topological insulators. For compact space-time manifolds by choosing $\theta = \pm(2n+1)\pi$; $n = 0, 1, \dots$, (1) describes the main features of the time reversal invariant topological insulators in 3 + 1 dimensions [2]. In particle physics (1) is known as the topological term for the θ -vacuum [3, 4].

One of the models which gives rise to the action (1) is obtained through a dimensional reduction from the 4 + 1 dimensional Dirac theory [2]. One considers the relativistic electrons interacting with the external gauge fields A_μ , as well as with the scalar field $\Theta(x)$ described by

$$\mathcal{L}_{3+1}(\Psi, \bar{\Psi}, A) = \bar{\Psi}[i\gamma^\mu(\partial_\mu - ieA_\mu) + \gamma^4(k_4 + \Theta) - m]\Psi. \quad (2)$$

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Here, k_4 is a constant and we consider electrons of charge $e > 0$. γ_4 is one of the gamma matrices introduced in $4 + 1$ dimensions and $\Theta(x)$ is the reminiscent of the gauge field $A_4(x)$. Integrating out the fermionic fields $\Psi, \bar{\Psi}$ in the path integral of (2), yields the effective action [2, 5]

$$S_A = \frac{e^2}{8\pi^2} \int d^4x \Theta(x) \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma. \quad (3)$$

$\Theta(x)$ is known in particle physics as axion field (for a review see [6]). For a uniform and constant axion field $\Theta = \theta$, (3) leads to (1).

The topological field theory action (1) is also related to the phase of mass term of quarks through the axial anomaly [7] (for a recent discussion see [8]). One deals with the fermionic fields with negative mass described by the Lagrangian density

$$\mathcal{L}_{MD} = \bar{\psi}_M [i\gamma^\mu (\partial_\mu - ieA_\mu) - \exp(-i\pi)M] \psi_M.$$

In terms of axial transformations one can obtain a positive mass term. Thus, integrating out the $\bar{\psi}_M, \psi_M$ fields in the path integral, with an appropriate renormalization procedure, one obtains the action (1) for $\theta = \pi$.

We would like to present another strictly $3 + 1$ dimensional model which leads to (1). We will introduce an action of neutral fermionic quasiparticles and show that by integrating out these fermionic fields one can obtain (1) through a renormalization procedure. These neutral fermions are not the fundamental particles but arise as excitations made of particle and hole (antiparticle) solutions of the underlying Dirac theory of electrons. On the same footing we can discuss the fractional topological insulators [9]. It will be shown that our approach can also be generalized to derive the BF type theory which was proposed to describe topological insulators in [10].

How to construct these neutral fermions from the underlying charged particles is not known. However, neutral fermion excitations have already been appeared in the study of the Hall effect for even filling fractions [11, 12]. Moreover, neutral Dirac fermions composed of two Majorana fermions arise in systems composed of topological, magnetic and superconducting insulators [13].

2 Effective field theory of topological insulators

In topological insulators the physical constituents are electrons denoted ψ_e . They interact with external electromagnetic fields according to the Dirac Lagrangian density

$$\mathcal{L}_D = \bar{\psi}_e [i\gamma^\mu (\partial_\mu - ieA_\mu) - m] \psi_e. \quad (4)$$

On-shell states satisfy the Dirac equation. As is well known, there are negative energy solutions of the Dirac equation describing positive charged particles which can be interpreted as holes. Hence, when we deal with the excitations of the Dirac particles we may consider particles of both sign. They can be tight together to form neutral degrees of freedom. Let us assume that in the bulk of topological insulator effectively there are neutral fermions. This guarantees that there is no electrical transport in the bulk, which is the basic property of an insulator. Obviously, neutral fermions couple to external electromagnetic fields due to their electric and magnetic dipole moments whose interaction terms covariantly given by $\sigma^{\mu\nu} F_{\mu\nu}$ and $i\gamma^5 \sigma^{\mu\nu} F_{\mu\nu}$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu; \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]; \quad \gamma_5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3.$$

We propose that inside a topological insulator these neutral fermions are described by the Lagrangian density

$$\mathcal{L}(\bar{\psi}, \psi, F) = \bar{\psi} \left[i\gamma^\mu \partial_\mu + \frac{1}{2} e\alpha (1 - i\gamma^5) \sigma^{\mu\nu} F_{\mu\nu} - m \right] \psi, \quad (5)$$

where α is a constant. In the related path integral one can integrate out the fermions to derive the effective action of the electromagnetic field strengths $S[F]$:

$$\exp(iS[F]) \equiv \int d\bar{\psi} d\psi \exp \left(i \int d^4x \mathcal{L}(\bar{\psi}, \psi, F) \right).$$

Formally, the effective action can be written as

$$S[F] = -i \ln \det [i\gamma^\mu \partial_\mu + \frac{1}{2}e\alpha(1 - i\gamma^5)\sigma^{\mu\nu} F_{\mu\nu} - m]. \quad (6)$$

As usual, it is preferable to work in the momentum space. One of the terms which (6) generates is

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} F_{\mu\nu}(k_1) F_{\rho\sigma}(k_2) \pi^{\mu\nu\rho\sigma}(k_1) \delta^4(k_1 - k_2). \quad (7)$$

We deal with the low energy limit where momentum of the external legs of the related Feynman diagram vanishes. In this limit (7) can be taken as the effective action. At the one loop level $\pi^{\mu\nu\rho\sigma}(k)$ can be written as

$$\pi^{\mu\nu\rho\sigma}(k) = \frac{i}{2}e^2 \int \frac{d^4p}{(2\pi)^4} \text{tr} \left[\left(\frac{\alpha}{2}(1 - i\gamma^5)\sigma^{\mu\nu} S_F(p) \right) \left(\frac{\alpha}{2}(1 - i\gamma^5)\sigma^{\rho\sigma} S_F(p - k) \right) \right], \quad (8)$$

where the Feynman propagator is defined as

$$S_F(p) = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2}.$$

Using the properties of Dirac matrices in $3 + 1$ dimensions, one can show that (8) reduces to two terms:

$$\pi^{\mu\nu\rho\sigma}(k) = \Sigma^{\mu\nu\rho\sigma}(k) + \Pi^{\mu\nu\rho\sigma}(k), \quad (9)$$

which are defined as

$$\begin{aligned} \Sigma^{\mu\nu\rho\sigma}(k) &= \frac{i}{16}e^2\alpha^2 \int \frac{d^4p}{(2\pi)^4} \frac{p_\alpha(p - k)_\beta \text{tr}\{[\gamma^\mu, \gamma^\nu]\gamma^\alpha[\gamma^\rho, \gamma^\sigma]\gamma^\beta\}}{(p^2 - m^2)((p - k)^2 - m^2)}, \\ \Pi^{\mu\nu\rho\sigma}(k) &= \frac{1}{16}e^2\alpha^2 m^2 \int \frac{d^4p}{(2\pi)^4} \frac{\text{tr}\{[\gamma^\mu, \gamma^\nu][\gamma^\rho, \gamma^\sigma]\gamma^5\}}{(p^2 - m^2)((p - k)^2 - m^2)}. \end{aligned}$$

$\Sigma^{\mu\nu\rho\sigma}(k)$ is quadratically divergent. In the low energy limit ($k \rightarrow 0$) it can be regularized within the dimensional regularization scheme. Indeed, in $4 - \epsilon$ dimensions we can perform the related integral as [14]

$$\int \frac{d^4p}{(2\pi)^4} \frac{p_\alpha p_\beta}{(p^2 - m^2)^2} = \frac{-i}{(4\pi)^{2-\epsilon/2}} \frac{\eta_{\alpha\beta}}{2} \Gamma(-1 + \epsilon/2) \left(\frac{1}{m^2}\right)^{-1+\epsilon/2},$$

where $\eta_{\alpha\beta}$ is the metric tensor and $\Gamma(x)$ is the gamma function. However, the multiplicative trace term can be seen to vanish:

$$\eta_{\alpha\beta} \text{tr}\{[\gamma^\mu, \gamma^\nu]\gamma^\alpha[\gamma^\rho, \gamma^\sigma]\gamma^\beta\} = 0.$$

Thus after regularization we set

$$\Sigma^{\mu\nu\rho\sigma}(0) = 0.$$

The other term in (9) is topological, i.e. it is independent of the space-time metric $\eta_{\mu\nu}$. In terms of the totally antisymmetric tensor $\epsilon^{\mu\nu\rho\sigma}$ originated from

$$\text{tr}\{\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\} = -4i\epsilon^{\mu\nu\rho\sigma},$$

it can be expressed as

$$\Pi^{\mu\nu\rho\sigma}(k) = e^2 m^2 \alpha^2 \epsilon^{\mu\nu\rho\sigma} I(k).$$

We defined

$$I(k) = -i \int \frac{d^4 p}{(2\pi)^4} \frac{1}{((p^2 - m^2)((p - k)^2 - m^2))}, \quad (10)$$

whose integrand is spherically symmetric in the low energy limit ($k \rightarrow 0$). The integral is logarithmically divergent which can be dimensionally regularized. To calculate (10) in d dimensions we introduce the parameter μ having the dimension of mass such that the integral

$$I(0) = -i \mu^{4-d} \int \frac{d^d p}{(2\pi)^4} \frac{1}{(p^2 - m^2)^2}, \quad (11)$$

remains dimensionless. Calculation of $I(0)$ in d dimensions leads to [14]

$$I(0) = \frac{1}{4\pi^{d/2}} \Gamma(2 - d/2) \left(\frac{\mu^2}{m^2}\right)^{2-d/2}.$$

Setting $d = 4 - \varepsilon$ the finite and infinite parts in the $\varepsilon \rightarrow 0$ limit, are separated as

$$I(0) = \frac{1}{(4\pi)^2} \left(\frac{2}{\varepsilon} + \ln\left(\frac{\mu^2}{m^2}\right) + \ln(4\pi) - \gamma + \dots \right).$$

γ is the Euler-Mascheroni constant. The bare coupling constant α , or the mass m , can be renormalized with the divergent constant Z to introduce the finite parameter θ_F as

$$Z m^2 \alpha^2 I(0) = \frac{\theta_F}{32\pi^2}.$$

As it is usual in the renormalization theory, the value of θ_F will be fixed in terms of the “experimentally measured quantities.” We conclude that

$$\pi^{\mu\nu\rho\sigma}(0) = \frac{e^2 \theta_F}{32\pi^2} \epsilon^{\mu\nu\rho\sigma}.$$

Therefore, in the low energy limit the renormalized effective action is

$$S[F] = \frac{e^2 \theta_F}{32\pi^2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x). \quad (12)$$

It is the same with the action (1) for $\theta_F = \theta$.

Properties of the space-time on which the theory is defined should be dictated by the physical arguments. We assume that the neutral quasiparticle of the initial Lagrangian density (5) is composed of the original particles and holes. For topological insulators the physical particles are the electrons minimally coupled to the gauge fields A_μ as given in (4). Then, there are some configurations which are consistent with periodic space-time yielding the quantization condition

$$\frac{e^2}{32\pi^2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = N, \quad (13)$$

where N is an integer. Thus the partition functions for $\theta_F = \pm(2n+1)\pi$; $n = 0, 1, \dots$, are the same, so that the theory is time reversal invariant.

Obviously, there is no a priori given condition for the space-time structure of the topological field theory action (12). We may choose the space-time to be periodic but yielding the quantization condition

$$\frac{e^2}{32\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = N_f^2 N, \quad (14)$$

where N_f is an odd integer. Then, we set $\theta_F = \pm(2n+1)\pi/N_f$; $n = 0, 1, \dots$, which defines the time reversal invariant fractional topological insulator [9, 15, 16]. In this case, as in [9], the partons denoted ψ_p , whose electromagnetic interaction Lagrangian density is given by

$$\mathcal{L}_p = \bar{\psi}_p \left[i\gamma^\mu (\partial_\mu - i\frac{e}{N_f} A_\mu) - m \right] \psi_p,$$

can be chosen as the physical particles in the bulk of the fractional topological insulator. They possess the electric charge e/N_f , so that the quantization condition (14) is justified.

3 BF theory description of topological insulators

Another effective field theory approach to describe the $3+1$ dimensional time reversal invariant topological insulators was proposed in [10]. In addition to the electromagnetic gauge field A_μ , they introduced the auxiliary fields a_μ , $b_{\rho\sigma}$ which are Abelian vector and antisymmetric tensor fields. It is argued that the time reversal invariant $3+1$ dimensional topological insulator is described by the BF type effective field theory given by

$$\mathcal{L}_{BF} = \frac{e}{2\pi} \epsilon^{\mu\nu\rho\sigma} a_\mu \partial_\nu b_{\rho\sigma} + \frac{e}{2\pi} \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu b_{\rho\sigma} + C \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_\nu \partial_\rho A_\sigma, \quad (15)$$

where C is a constant parameter. To fix the value of it as $C = \pm e^2/8\pi$, the gauge charge lattice was specified as an additional information. Thus, integrating out a_μ and $b_{\rho\sigma}$ fields in the related path integral yields

$$S_{3D} = \pm \frac{e^2}{8\pi} \int d^4x \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma, \quad (16)$$

which is the same with (1) for $\theta = \pm 1$.

It is possible to construct (15) in terms of the procedure outlined in Section 2. For this purpose let us introduce the neutral fermions denoted ψ_A ; $A = 1, 2, \dots, 6$ and define the Lagrangian density

$$\begin{aligned} \mathcal{L} = & \bar{\psi}_1 \left[i\gamma^\mu \partial_\mu + \frac{1}{2} \lambda (1 - i\gamma^5) \sigma^{\mu\nu} (F_{\mu\nu} + b_{\mu\nu}) - m \right] \psi_1 \\ & + \bar{\psi}_2 \left[i\gamma^\mu \partial_\mu + \frac{1}{2} \lambda (1 + i\gamma^5) \sigma^{\mu\nu} (F_{\mu\nu} - b_{\mu\nu}) - m \right] \psi_2 \\ & + \bar{\psi}_3 \left[i\gamma^\mu \partial_\mu + \frac{1}{4} \beta (1 - i\gamma^5) \sigma^{\mu\nu} (F_{\mu\nu} + f_{\mu\nu}) - m \right] \psi_3 \\ & + \bar{\psi}_4 \left[i\gamma^\mu \partial_\mu + \frac{1}{4} \beta (1 + i\gamma^5) \sigma^{\mu\nu} (F_{\mu\nu} - f_{\mu\nu}) - m \right] \psi_4 \\ & + \bar{\psi}_5 \left[i\gamma^\mu \partial_\mu + \frac{1}{2} \lambda (1 - i\gamma^5) \sigma^{\mu\nu} (f_{\mu\nu} + b_{\mu\nu}) - m \right] \psi_5 \\ & + \bar{\psi}_6 \left[i\gamma^\mu \partial_\mu + \frac{1}{2} \lambda (1 + i\gamma^5) \sigma^{\mu\nu} (f_{\mu\nu} - b_{\mu\nu}) - m \right] \psi_6, \end{aligned}$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. Now, we can integrate out each $\psi_A, \bar{\psi}_A$ neutral fermion degrees of freedom following the procedure of Section 2. Obviously, at the one loop level integrals are divergent which can be expressed in the low energy limit as $I(0)$ which is defined in (11). The bare coupling constants λ and β can be renormalized to define the finite constants Λ_F and C_F as

$$Zm^2\lambda^2 I(0) = \Lambda_F, \quad Zm^2\beta^2 I(0) = C_F.$$

Then, we can write the low energy effective action as

$$S[A, b, a] = \int d^4x \{ \Lambda_F [\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu b_{\rho\sigma} + \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_\nu b_{\rho\sigma}] + C_F \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho a_\sigma \}. \quad (17)$$

As usual the renormalized quantities should be fixed by some additional conditions. We can choose $\Lambda_F = 1/2\pi$ and $C_F = \pm e^2/8\pi$, so that (17) yields the BF theory described by (15), up to surface terms. Obviously, integrating out first $b_{\mu\nu}$ then a_μ fields leads to the action (16).

4 Discussions

We constructed in the low energy limit, the effective action of $3+1$ dimensional topological insulators (12), by integrating out the neutral Dirac particles described by the Lagrangian density (5). We assume that the neutral fermions are formed as excitations of the original physical particles which are electrons or partons. Our procedure of calculating the effective topological field theory (12) does not refer to the origin of the neutral fermions. The original physical particles establish the features of the space-time manifold on which the effective topological field theory is defined. Then, according to the quantization conditions like (13) and (14), the value of the physical coupling constant θ_F should be chosen.

Interchanging Abelian field strengths with non-Abelian ones in the Lagrangian density (5) will not alter the construction of the low energy effective action (12) presented in Section 2. Therefore, our procedure is also valid for non-Abelian gauge fields. In this case, quantization condition of the effective action will be changed [17], so that the renormalized coefficient θ_F should be appropriately chosen.

Although, mechanism of constructing neutral fermions from the original ones is an open issue, our procedure indicates that different kind of topological insulators can be studied in a unified manner.

References

- [1] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
- [2] X-L. Qi, T. L. Hughes, and S-C. Zhang, Phys. Rev. B 78 (2008) 195424.
- [3] C. G. Callan, R. F. Dashen and D. J. Gross, Phys. Lett. B 63 (1976) 334.
- [4] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37 (1976) 172.
- [5] Ö.F. Dayi, M. Elbistan and E. Yunt, Annals of Physics 327 (2012) 935.
- [6] J. E. Kim, Phys. Rep. 150 (1977) 1.
- [7] V. Baluni, Phys. Rev. D 19 (1979) 7.
- [8] C. Hoyos, K. Jensen, and A. Karch, Phys. Rev. D 82 (2010) 086001.
- [9] J. Maciejko, X.L. Qi, A. Karch, and S.C. Zhang, Phys. Rev. Lett. 105 (2010) 246809.
- [10] G. Y. Cho, and J. E. Moore, Annals of Physics 326 (2011) 1515.
- [11] G. Moore and N. Read, Nucl. Phys. B 360 (1991) 362.
- [12] M. Greiter, X. G. Wen, and F. Wilczek, Phys. Rev. Lett. 66 (1991) 3205.
- [13] L. Fu and C. L. Kane, Phys. Rev. Lett. 102 (2009) 216403.
- [14] M. E. Peskin, and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Westview Press, Boulder, 1995).
- [15] B. Swingle, M. Barkeshli, M. McGreevy, and T. Senthil, Phys. Rev. B 83 (2011) 195139.
- [16] K.-S. Park and H. Han, Phys. Rev. B 82 (2010) 153101.
- [17] E. Witten, Selecta Math. 1 (1995) 383, hep-th/9505186.